Special topics in analysis

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The goal of this course is to develop the theory of pointwise differentiation in the simplest case of real valued functions defined on Euclidean space. A function is k times pointwise differentiable at a point if and only if it may be approximated by a polynomial function of degree at most k to k-th order at that point; in particular, derivatives of order $k-1$ need not exist in a neighbourhood of the point. Such derivatives for instance occur in the study of Sobolev functions, convex functions ($k = 2$), and viscosity solutions to fully nonlinear elliptic equations of second order ($k = 2$). Our motivation however is mainly from geometry: the theory presented acts both as a model case and a toolbox for recent characterisations of rectifiability of order k which essentially is the weakest possible sense in which a set (or, a function) may be considered to have k-th order differentials almost everywhere.

Some knowledge of linear algebra, first order differentials, measure, and Lebesgue integration is prerequisite for this course, but the treatment is self-contained with regard to all those topics in descriptive set theory, multilinear algebra, and higher order differentiation, that occur. A particular feature of our approach is the development of the concept of symmetric algebra of a vectorspace to effectively treat polynomial functions. The topics covered are as follows.

(1) Descriptive set theory: Lipschitzian maps, Suslin sets, Borel functions (see [Fed69, 2.2.7–2.2.14]).

(2) Multilinear algebra: tensor algebras, graded algebras, the symmetric algebra of a vectorspace, symmetric forms, and polynomial functions (see [Fed69, Sections 1.2, 1.3, 1.9, 1.10]).

(3) Higher order differentiation: higher differentials, partitions of unity, differentiable extensions of functions (see [Fed69, 3.1.11–3.1.15]).

(4) Approximate differentiation and rectifiability of higher order (see [Isa87]).

(5) Pointwise differentiation of higher order (see [Men19, Sections 2, 4]).

References

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